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ON FACTORIZATION OF MATRIX POLYNOMIAL WITH RESPECT TO THE UNIT CIRCLE

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Abstract. The problem of factorization of the second order matrix polynomial with respect to the unit circle is considered. It is shown, that in the case when the roots lay on the unit circle, the solution for the factorization problem can be obtained using the Bass relation. To demonstrate the possibility of non-uniqueness of the solution of factorization problem an example is included.

Keywords: factorization, unit circle, matrix polynomial, Bass relation.

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1. Introduction

Algorithms for solution of various linear time invariant control systems synthesis problems require factorization of matrix polynomials [1, 4]. In this context, various algorithms of factorization were proposed, in particular, factorization algorithms with respect to the unit circle [2, 3, 6, 7, 10, 12, 13]. It can be stated that the development of algorithms for the factorization of matrix polynomials remains relevant at the present time (see [9], where there are further references). Thus, in [9] the problem of factorization of the matrix polynomial

$$\varphi(z) = z^{-1}A_{-1} + A_0 + zA_1, \tag{1}$$

with respect to the unit circle is considered without assumptions about the symmetric arrangement of its roots. In (1), A_{-1}, A_0, A_1 are the given matrices of the dimension $n \times n$. It is shown in [9] that the polynomial (1) can be factorized, i.e. presented in the form

$$\varphi(z) = (I - zR)K(I - z^{-1}G). \tag{2}$$

Hereinafter I is the unit matrix of corresponding dimension. The matrices G and R satisfy the following unilateral quadratic matrix equations:

$$A_1G^2 + A_0G + A_{-1} = 0, \tag{3}$$

$$R^2A_{-1} + RA_0 + A_1 = 0. \tag{4}$$

We note that, according to [9], the factorization (5) is called canonical if $\rho(R) < 1, \ \rho(G) < 1$ and it is called weakly canonical if $\rho(R) \leq 1, \ \rho(G) \leq 1$. Here $\rho(\cdot)$ denotes the spectral radius.

Comparing the coefficients of the corresponding degrees of z in (1), (2), we can obtain the following relation:

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$$K + RKG = A_0, -KG = A_{-1}, -RK = A_1. \tag{5}$$

From these relations it follows that

$$K = A_0 + A_1 G, K = A_0 + RA_{-1}, \tag{6}$$

$$A_1 G = RA_{-1}. \tag{7}$$

Substituting (7) into (3), (4) we obtain

$$R(A_1 G + A_0) + A_1 = 0, \tag{8}$$

$$RA_{-1}G + A_0 G + A_{-1} = 0. \tag{9}$$

Thus, determining the matrix G by solving (3), we can find the matrices K, R from the linear relations (6), (8), (9). In the case when the matrices G, R, K are given (see example), the relation (5) can be used to calculate the matrix coefficients A_1, A_0, A_{-1} of the polynomial $\varphi(z)$. Note that the factorization (2) is called the left factorization of the polynomial (1) (see [4]). To find the right factorization of the polynomial (1), we can proceed as follows. Using the relation (2), we find the factorization of the polynomial

$$\tilde{\varphi}(z) = z^{-1}A'_{-1} + A'_0 + zA'_1. \tag{10}$$

In (10) and in what follows, $(')$ denotes transposition. We represent the polynomial $\tilde{\varphi}(z)$ in the form (2):

$$\tilde{\varphi}(z) = (I - z\tilde{R})\tilde{K}(I - z^{-1}\tilde{G}).$$

Taking into account that $\tilde{\varphi}(z)' = \varphi(z)$, we obtain the following relation:

$$\varphi(z) = \tilde{\varphi}'(z) = (I - z^{-1}\tilde{G}')\tilde{K}'(I - z\tilde{R}'), \tag{11}$$

which determines the right factorization of the polynomial (1) (see, example).

We note that in [9] much attention is paid to the case when among the roots of a polynomial there are roots lying on the unit circle. This can worsen the convergence of the computational process of finding the solution of equations (3), (4). In this connection, in [9] the procedures are considered which make it possible to transform the original equation in such a way that the roots whose modulus is equal to one, are excluded.

Below we show the possibility of using the algorithm [5] to find the solution of equation (3) and, as a consequence, the solution of the factorization problem of the polynomial (1), in the case of roots whose modulus is one (see the example in the sequel).

2. Unilateral quadratic matrix equation [1]

It is known that various engineering problems are related to the theory of oscillations. Here we should note the theory of strongly damped systems [11], in which the central place is occupied by the questions of finding the roots of the matrix (or operator [11]) equation

$$A_2 X^2 + A_1 X + A_0 = 0. \tag{12}$$

In [8], the matrix equation (12) is called the unilateral quadratic matrix equation (UQME). It is obvious that it coincides with (3) with the accuracy on the notations.

We rewrite equation (12) in the following form

$$M_1 \begin{bmatrix} I \\ X \end{bmatrix} = F_1 \begin{bmatrix} I \\ X \end{bmatrix} B, \tag{13}$$

$$M_1 = \begin{bmatrix} 0 & I \\ -A_0 & -A_1 \end{bmatrix}, F_1 = \begin{bmatrix} I & 0 \\ 0 & A_2 \end{bmatrix}, B = X.$$

The problem is to construct a procedure that allows us to transform (13) to the form

$$M_p \begin{bmatrix} I \\ X \end{bmatrix} = F_p \begin{bmatrix} I \\ X \end{bmatrix} \Pi(B), \tag{14}$$

where $\Pi(B)$ is the some polynomial from the matrix B. Obviously, if $\Pi(B) = 0$ (for example, $\Pi(B)$ is the characteristic polynomial of the matrix B), then the relation (14) turns into the following linear equation with respect to X:

$$M_p \begin{bmatrix} I \\ X \end{bmatrix} = 0, \tag{15}$$

which can be rewriten in the form

$$M_{p2} X = -M_{p1},$$

if we split the matrix M_p into blocks: $M_p = \begin{bmatrix} M_{p1} & M_{p2} \end{bmatrix}$. It is obvious that the relation (15) determines the solution X only if the matrix M_{p2} is a matrix of full rank. In other words, the proposed algorithm "works" only in cases when M_{p2} is a matrix of full rank.

We note that if the matrix F_1 is invertible, then transforming (13) to the form

$$H_f \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} B, H_f = F_1^{-1} M_1, \tag{16}$$

it is easy to obtain the relation (15), in which we have the matrix $M_p = \Pi(H_f)$. However, if the matrix F_1 is singular, we find another approach.

Without assuming the invertibility of the matrix A_2, the transformation (13) to the form (14) yields matrices M_2, F_2 in the relation

$$M_2 \begin{bmatrix} I \\ X \end{bmatrix} = F_2 \begin{bmatrix} I \\ X \end{bmatrix} B^2. \tag{17}$$

Multiplying (13) on the right by B we obtain

$$M_1 \begin{bmatrix} I \\ X \end{bmatrix} B = F_1 \begin{bmatrix} I \\ X \end{bmatrix} B^2. \tag{18}$$

Let us introduce the matrices D_1, L_1 which satisfy the following relation:

$$L_1 M_1 = D_1 F_1. \tag{19}$$

Multiplying the equations (13), (18) on the left by the D_1, L_1, respectively, we obtain

$$D_1 M_1 \begin{bmatrix} I \\ X \end{bmatrix} = L_1 F_1 \begin{bmatrix} I \\ X \end{bmatrix} B^2, \tag{20}$$

i.e. as the matrices M_2, F_2, appearing in (17), we can take the following matrices:

$$M_2 = D_1 M_1, F_2 = L_1 F_1. \tag{21}$$

Taking into account that according to (19) the matrix $\begin{bmatrix} L_1 & D_1 \end{bmatrix}'$ is the kernel of the matrix $\begin{bmatrix} M_1 \\ -F_1 \end{bmatrix}'$, one can find matrices L_1, D_1 in (21) using the procedure null.m of the package MATLAB.

A similar procedure can also be used to construct a relation in which higher degrees of the matrix B appear.

Let (13) be transformed to the form

$$M_k \begin{bmatrix} I \\ X \end{bmatrix} = F_k \begin{bmatrix} I \\ X \end{bmatrix} B^k. \tag{22}$$

We construct an analogous relation in which the matrix B has degree $k+1$. Let us multiply (22) from right by B and introduce the matrices $D_k,\ L_k$, which satisfy the relation

$$L_k M_k = D_k F_1. \tag{23}$$

We then have

$$D_k M_1 \begin{bmatrix} I \\ X \end{bmatrix} = L_k F_k \begin{bmatrix} I \\ X \end{bmatrix} B^{k+1},$$

i.e. $M_{k+1} = D_k M_1$, $F_{k+1} = L_k F_k$.

According to (23), the matrix $\begin{bmatrix} L_k & D_k \end{bmatrix}'$ is the kernel of the matrix $\begin{bmatrix} M_k \\ -F_1 \end{bmatrix}'$ and, therefore, as already noted, can be used the null.m procedure of the MATLAB package to find the matrices $D_k,\ L_k$.

Let us note that $F_{k+1} = L_k F_k$ and, therefore,

$$F_k = L_{k-1} \dots L_1 F_1. \tag{24}$$

Thus, we described a procedure that allows one to construct the matrices $M_k,\ F_k$, appearing in (22). We use it to determine M_p in (7), assuming that the polynomial $\Pi(B)$ appearing in (14) given by

$$\Pi(B) = \beta_0 B^m + \beta_1 B^{m-1} + \dots + \beta_{m-1} B + \beta_m I \tag{25}$$

is the characteristic polynomial of the matrix B, where $\beta_0 = 1$. For it as a first step, it is necessary in (22) to make equal the matrix coefficients for B^k for $k = 1, \dots, m$. For this purpose, taking into account (24), we multiply on the left each of the relations (22) by $L_m L_{m-1} \dots L_k$. We replenish these relations by the identity

$$F_m \begin{bmatrix} I \\ X \end{bmatrix} = F_m \begin{bmatrix} I \\ X \end{bmatrix}. \tag{26}$$

We multiply (26) by β_m , and the relations which in B^k appears, by β_{m-k} (the coefficients β_i are determined by (25)). Adding them, we get

$$(\beta_m F_m + \beta_{m-1} L_{m-1} \dots L_1 M_1 + \dots + \beta_1 L_{m-1} M_{m-1} + M_m) \begin{bmatrix} I \\ X \end{bmatrix} = F_m \begin{bmatrix} I \\ X \end{bmatrix} \Pi(B) = 0.$$

Consequently, the matrix M_p, appearing in (15) has the form

$$M_p = \beta_m F_m + \beta_{m-1} L_{m-1} \dots L_1 M_1 + \dots + \beta_1 L_{m-1} M_{m-1} + M_m. \tag{27}$$

Thus, the solution of the equation (12) is determined by the relation (15), in which the matrix M_p has the form (27). The coefficients β_i are determined by the (25), and the matrices M_1, F_1 by the relation (13).

Let us consider the procedure for determining the coefficients β_i in (25) (the characteristic polynomial of the matrix B). Since, $B = X$, then, it is obvious that the use of standard computational procedures, for example poly.m of the MATLAB package, is problematic for finding the coefficients β_i. Obviously, the eigenvalues of the matrix X belong to the spectrum of the pencil

$$M_1 - \lambda F_1, \tag{28}$$

which allows us to find the coefficients β_i by selecting the roots of (25) from the eigenvalues of the pencil (28) (see the examples).

3. Examples

Example 1 (Example 4.4 [4]). Matrices appearing in (1) have the form

$$A_{-1} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, A_0 = \begin{bmatrix} 1 & -1 \\ -1 & 5 \end{bmatrix}, A_1 = A'_{-1}.$$

The polynomial (1) must be represented in the form (11). The corresponding matrices M_1, F_1 in (13) have the form

$$M_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ -1 & 1 & 1 & -5 \end{bmatrix}, F_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Note that these matrices do not have inverses. The problem consists in finding the matrix $\tilde{G}$, defined by (3), i.e. in finding the solution of equation (12), in which the matrices are determined by the coefficients of the polynomial (10). In order to use the relations (15), the matrix M_p, in which (27) is defined, it is necessary, according to (21), to find the matrices M_2, F_2 . In turn, this requires finding the matrix N, which is the kernel of the matrix $\begin{bmatrix} M_1 \\ F_1 \end{bmatrix}'$. So, we have

$$N = \begin{bmatrix} 0.6492 & -0.0940 & -0.0157 & 0.0157 \\ -0.0494 & 0.9065 & 0.1845 & -0.1845 \\ 0.6971 & 0.1292 & -0.1824 & 0.1824 \\ 0.0480 & 0.2232 & -0.1667 & 0.1667 \\ -0.0480 & -0.2232 & 0.1667 & -0.1667 \\ 0.0480 & 0.2232 & -0.1667 & 0.1667 \\ 0.2040 & -0.0401 & 0.9178 & 0.0822 \\ -0.2040 & 0.0401 & 0.0822 & 0.9178 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0.2040 & -0.2040 & -0.4559 & 1.2717 \\ -0.0401 & 0.0401 & -0.1430 & -0.0940 \\ -0.0822 & 0.0822 & -0.6689 & 0.3401 \\ -0.9178 & 0.9178 & 0.6689 & -4.3401 \end{bmatrix}, F_2 = \begin{bmatrix} 0.6492 & -0.0494 & 0 & 0.6492 \\ -0.0940 & -0.9065 & 0 & -0.0940 \\ -0.0157 & 0.1845 & 0 & -0.0157 \\ 0.0157 & -0.1845 & 0 & 0.0157 \end{bmatrix}.$$

Appearing in (25) polynomial $\Pi(B)$ has the form $\Pi(B) = B^2$, i.e. $\beta_0 = 1$, and other coefficients β_1, β_2 are equal to zero. Therefore, according to (27), $M_p = M_2$ and, using (15) we obtain

$$\tilde{G} = X = \frac{1}{4} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Thus, according to (6), (8) we have

$$\tilde{K} = \begin{bmatrix} 0.75 & -0.75 \\ -0.75 & 4.75 \end{bmatrix}, \tilde{R} = \tilde{G}'.$$

Decomposing the matrix $\tilde{K}'$ to Cholesky factors,

$$\tilde{K}' = \dot{C}_k' C_k,$$

we obtain, according to (11), the following expression for $\varphi(z)$:

$$\varphi(z) = \Pi'(z^{-1})\Pi(z),$$

where

$$\Pi(z) = C_k(1 - zR') = \begin{bmatrix} 0.866 & -0.866 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1/2 & -1/2 \end{bmatrix} z.$$

This expression coincides with accuracy $\approx 10^{16}$ with the exact value given in [4].

Example 2. Let us show that the polynomial (1), generally speaking, can have more than one representation in the form (2). In (2), set

$$G = \begin{bmatrix} -1 & 0 \\ 5 & 0 \end{bmatrix}, R = \begin{bmatrix} 1 & 4 \\ 0 & 0.8 \end{bmatrix}, K = \frac{1}{100}\begin{bmatrix} 3 & 4 \\ 0 & 1 \end{bmatrix}. \quad (29)$$

To these initial data, according to (5), correspond the following matrices appearing in (1):

$$A_{-1} = \begin{bmatrix} -0.17 & 0 \\ -0.05 & 0 \end{bmatrix}, A_0 = \begin{bmatrix} 0.4 & 0.04 \\ 0.04 & 0.01 \end{bmatrix}, A_1 = \begin{bmatrix} -0.03 & -0.08 \\ 0 & -0.008 \end{bmatrix}, \quad (30)$$

and, according to (2), one of the possible factorizations of the polynomial (1):

$$\varphi(z) = (I - zR)K(I - z^{-1}G). \quad (31)$$

In (31) the matrices R, K, G are determined by (29).

Using the results of p. 2, we construct another factorization of the polynomial (1), different from (31), the matrices in which is defined by (30). With these initial data, the matricesM_1, F_1 in (13) have the form:

$$M_1 = \begin{bmatrix} 0 & I \\ -A_{-1} & -A_0 \end{bmatrix}, F_1 = \begin{bmatrix} I & 0 \\ 0 & A_1 \end{bmatrix}.$$

In this example, the matrix F_1 has an inverse. Therefore, to solve the factorization problem, we can use (16). The matrix $H_f = F_1^{-1}M_1$ has the following eigenvalues: 0; -1; 1; 1.25. Thus, as a matrix $\Pi(H_f)$, the following matrices can be chosen:

$$H_1 = H_f^2 + H_f, H_2 = H_f^2 - H_f.$$

If in (15) as the matrix M_p we choose the matrix H_1 ($M_p = H_1$), and as X the matrix G, which is determined by (29), then for such choice of these matrices the relation (15) will be satisfied. If we take matrix H_2 as the matrix M_p, then in the result of the solution of system (15) we obtain that $G_2 = X = \begin{bmatrix} 1 & 0 \\ 5 & 0 \end{bmatrix}$ and, respectively, according to (6), (8), we obtain $K_2 = \frac{1}{100}\begin{bmatrix} -3 & 4 \\ 0 & 1 \end{bmatrix}$, $R_2 = \begin{bmatrix} -1 & 12 \\ 0 & 0.8 \end{bmatrix}$. The found values G_2, K_2, R_2 according to (2), is determining another factorization of the polynomial (1) different from (31):

$$\varphi(z) = (I - zR_2)K_2(I - z^{-1}G_2).$$

This fact is related to the fact that the determinant of the polynomial (1) has two roots, modulus of which is 1.

4. Conclusion

The factorization problem with respect to the unit circle of the second-order matrix polynomial is considered in [9]. It is shown that in the case of roots lying on a unit circle, the algorithm [5] for solving of the unilateral second-order matrix equation based on the Bass relation can be used to solve the factorization problem. On example, the possibility of non-uniqueness of the solution of the factorization problem is shown.

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